

The enclosure method for inverse obstacle scattering problems with dynamical data over a finite time interval

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Abstract

A simple method for some class of inverse obstacle scattering problems is introduced. The observation data are given by a wave field measured on a known surface surrounding unknown obstacles over a *finite* time interval. The wave is generated by an initial data with compact support outside the surface. The method yields the distance from a given point outside the surface to obstacles and thus more than the convex hull.

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1 Introduction and statements of the results

The aim of this paper is to introduce a simple method for some class of inverse obstacle scattering problems in which some *dynamical data* over a *finite* time interval are used as the observation data.

In order to explain the essence of the idea we consider two inverse obstacle scattering problems whose governing equations are given by the wave equations in three dimensions.

1.1 Sound hard obstacles

Let $0 < T < \infty$. Let $D \subset \mathbf{R}^3$ be a bounded open set with smooth boundary such that $\mathbf{R}^3 \setminus \overline{D}$ is connected. Given $f \in L^2(\mathbf{R}^3)$ with compact support satisfying $\text{supp } f \cap \overline{D} = \emptyset$

let $u = u(x, t)$ satisfy the initial boundary value problem:

$$\begin{aligned}
\partial_t^2 u - \Delta u &= 0 \text{ in } (\mathbf{R}^3 \setminus \overline{D}) \times]0, T[, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial D \times]0, T[, \\
u(x, 0) &= 0 \text{ in } \mathbf{R}^3 \setminus \overline{D}, \\
\partial_t u(x, 0) &= f(x) \text{ in } \mathbf{R}^3 \setminus \overline{D}.
\end{aligned} \tag{1.1}$$

Here we denote the unit outward normal to ∂D by the symbol ν .

Let Ω be a bounded domain with smooth boundary such that $\overline{D} \subset \Omega$ and $\mathbf{R}^3 \setminus \overline{\Omega}$ is connected. We denote the unit outward normal to $\partial\Omega$ by ν again. The $\partial\Omega$ is considered as the location of the receivers of the acoustic wave produced by an emitter located at the support of f . In this paper first we consider the following problem.

Inverse Problem I. Assume that D is unknown. Extract information about the location and shape of D from u on $\partial\Omega \times]0, T[$ for some fixed *known* f satisfying $\text{supp } f \cap \overline{\Omega} = \emptyset$ and $T < \infty$.

This is a quite natural problem, however, to my best knowledge, it seems that no attempt has been done. Clearly the main obstruction is the *finiteness* of T and f is fixed.

Note that u in $(\mathbf{R}^3 \setminus \overline{\Omega}) \times]0, T[$ can be computed from u on $\partial\Omega \times]0, T[$ by the formula

$$u = z \text{ in } (\mathbf{R}^3 \setminus \overline{\Omega}) \times]0, T[\tag{1.2}$$

where z solves the initial boundary value problem in $\mathbf{R}^3 \setminus \overline{\Omega}$:

$$\begin{aligned}
\partial_t^2 z - \Delta z &= 0 \text{ in } (\mathbf{R}^3 \setminus \overline{\Omega}) \times]0, T[, \\
z &= u \text{ on } \partial\Omega \times]0, T[, \\
z(x, 0) &= 0 \text{ in } \mathbf{R}^3 \setminus \overline{\Omega}, \\
\partial_t z(x, 0) &= f(x) \text{ in } \mathbf{R}^3 \setminus \overline{\Omega}.
\end{aligned} \tag{1.3}$$

Thus the problem can be reformulated as

Inverse Problem I'. Extract information about the location and shape of D from u in $(\mathbf{R}^3 \setminus \overline{\Omega}) \times]0, T[$ for some known f satisfying $\text{supp } f \cap \overline{\Omega} = \emptyset$ and $T < \infty$.

Now we state the result. Let B be an open ball with $\overline{B} \cap \overline{\Omega} = \emptyset$. Choose the initial data $f \in L^2(\mathbf{R}^3)$ in such a way that:

$$(I1) \quad f(x) = 0 \text{ a.e. } x \in \mathbf{R}^3 \setminus B,$$

$$(I2) \quad \text{there exists a positive constant } C \text{ such that } f(x) \geq C \text{ a.e. } x \in B \text{ or } -f(x) \geq C \text{ a.e. } x \in B.$$

Let $\tau > 0$ and $v \in H^1(\mathbf{R}^3)$ be the weak solution of

$$(\Delta - \tau^2)v + f(x) = 0 \text{ in } \mathbf{R}^3. \quad (1.4)$$

This means that v satisfies

$$-\int_{\mathbf{R}^3} \nabla v \cdot \nabla \varphi dx - \tau^2 \int_{\mathbf{R}^3} v \varphi dx = -\int_{\mathbf{R}^3} f \varphi dx, \quad \forall \varphi \in H^1(\mathbf{R}^3). \quad (1.5)$$

The v is unique and is given by the explicit form

$$v(x; \tau) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{e^{-\tau|x-y|}}{|x-y|} f(y) dy, \quad x \in \mathbf{R}^3. \quad (1.6)$$

In this paper given two sets A and B we denote by $\text{dist}(A, B)$ the distance between A and B :

$$\text{dist}(A, B) = \inf\{|x - y| \mid x \in A, y \in B\}.$$

If A or B consists of a single point, say $B = \{p\}$, we write $\text{dist}(A, B) = d_A(p)$.

Set

$$w(x; \tau) = \int_0^T e^{-\tau t} u(x, t) dt, \quad x \in \mathbf{R}^3 \setminus \overline{\Omega}, \quad \tau > 0. \quad (1.7)$$

Our result is the following extraction formula from w and $\partial w / \partial \nu$ on $\partial\Omega \times]0, T[$ which can be computed from the data u in $(\mathbf{R}^3 \setminus \overline{\Omega}) \times]0, T[$.

Theorem 1.1. *If the observation time T satisfies*

$$T > 2\text{dist}(D, B) - \text{dist}(\Omega, B), \quad (1.8)$$

then there exists a $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS > 0$$

and the formula

$$\lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \log \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS = -\text{dist}(D, B), \quad (1.9)$$

is valid.

Since $\text{dist}(D, B) + \sqrt{|\partial B|/4\pi}$ coincides with the distance from the center of B to D , (1.9) yields the information about $d_D(p)$ for a given point p in $\mathbf{R}^3 \setminus \overline{\Omega}$. Therefore one can extract more than the convex hull of D . Note that we do not assume the special form of f except for the conditions (I1) and (I2).

The restriction (1.8) on the observation time T is reasonable. Define the quantity

$$l(\partial B, \partial D, \partial\Omega) = \inf \{|x - y| + |y - z| \mid x \in \partial B, y \in \partial D, z \in \partial\Omega\}.$$

This is the minimum length of the broken paths that start at $x \in \partial B$ and reflect at $y \in \partial D$ and return to $z \in \partial\Omega$. We have

Proposition 1.1.

$$2\text{dist}(D, B) - \text{dist}(\Omega, B) \geq l(\partial B, \partial D, \partial \Omega).$$

Proof. One can find $x_0 \in \partial B$ and $y_0 \in \partial D$ such that $|x_0 - y_0| = \text{dist}(D, B)$. Let $l(x_0, y_0) = \{tx_0 + (1-t)y_0 \mid 0 < t < 1\}$. We see that $l(x_0, y_0) \cap \partial \Omega \neq \emptyset$. Let $z_0 \in l(x_0, y_0) \cap \partial \Omega$. We have $|x_0 - z_0| \geq \text{dist}(\Omega, B)$. Thus $2\text{dist}(D, B) - \text{dist}(\Omega, B) \geq 2|x_0 - y_0| - |x_0 - z_0|$. Since $|x_0 - z_0| = |x_0 - y_0| - |y_0 - z_0|$ we have $2\text{dist}(D, B) - \text{dist}(\Omega, B) \geq |x_0 - y_0| + |y_0 - z_0|$. \square

Therefore (1.8) ensures that $T > l(\partial B, \partial D, \partial \Omega)$. This means that T is greater than the *first arrival time* of a signal with the unit propagation speed that starts at a point on ∂B at $t = 0$, reflects at a point on ∂D and goes to a point on $\partial \Omega$. However, curiously enough in the proof of Theorem 1.1 we never make use of the *finite propagation property* of the signal governed by the wave equation.

The procedure of extracting information about the location of D is extremely simple and summarized as follows.

- (i) Give an open ball B with $\overline{B} \cap \overline{\Omega} = \emptyset$. Using the initial data f satisfying (I1) and (I2) generate the wave field u .
- (ii) Choose a large T , say, such that $T > 2 \sup \{|y - x| \mid y \in \Omega, x \in B\} - \text{dist}(\Omega, B)$ and measure u on $\partial \Omega$ over the time interval $]0, T[$.
- (iii) Compute the values of z in a neighbourhood of $\partial \Omega$ relative to $\mathbf{R}^3 \setminus \overline{\Omega}$ over $]0, T[$ by solving (1.3).
- (iv) Choose a large τ and compute w and $\partial w / \partial \nu$ on $\partial \Omega$ over the time interval $]0, T[$ via (1.2) and (1.7).
- (v) Compute v and $\partial v / \partial \nu$ on $\partial \Omega$ via (1.6).
- (vi) Compute the quantity

$$\frac{1}{2\tau} \log \int_{\partial \Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS$$

as an approximation of $-\text{dist}(D, B)$.

One choice of f gives one information about D by the procedure (i) to (vi). This means that we don't need to use *many* f s to get $d_D(p)$ for a *single* p . This is the decisive character of our procedure.

1.2 Penetrable obstacles

Given $f \in L^2(\mathbf{R}^3)$ with compact support let $u = u(x, t)$ satisfy the initial value problem:

$$\begin{aligned} \partial_t^2 u - \nabla \cdot \gamma \nabla u &= 0 \text{ in } \mathbf{R}^3 \times]0, T[, \\ u(x, 0) &= 0 \text{ in } \mathbf{R}^3, \\ \partial_t u(x, 0) &= f(x) \text{ in } \mathbf{R}^3, \end{aligned} \tag{1.10}$$

where $\gamma = \gamma(x) = (\gamma_{ij}(x))$ satisfies

- for each $i, j = 1, 2, 3$ $\gamma_{ij}(x) = \gamma_{ji}(x) \in L^\infty(\mathbf{R}^3)$;
- there exists a positive constant C such that $\gamma(x)\xi \cdot \xi \geq C|\xi|^2$ for all $\xi \in \mathbf{R}^3$ and a. e. $x \in \mathbf{R}^3$.

This subsection is concerned with the extraction of information about *discontinuity* of γ from u on $\partial\Omega \times]0, T[$ for some f for a fixed $T < \infty$. However, we do not consider the completely general case. Instead we assume:

- there exists a bounded open set D with a smooth boundary such that $\gamma(x)$ a.e. $x \in \mathbf{R}^3 \setminus D$ coincides with the 3×3 identity matrix I_3 .

Write $h(x) = \gamma(x) - I_3$ a.e. $x \in D$. Our second inverse problem is the following.

Inverse Problem II. Assume that both D and h are *unknown* and that one of the following two conditions is satisfied:

- (A1) there exists a positive constant C such that $-h(x)\xi \cdot \xi \geq |\xi|^2$ for all $\xi \in \mathbf{R}^3$ and a.e. $x \in D$;
- (A2) there exists a positive constant C such that $h(x)\xi \cdot \xi \geq |\xi|^2$ for all $\xi \in \mathbf{R}^3$ and a.e. $x \in D$.

Extract information about the location and shape of D from u on $\partial\Omega \times]0, T[$ for some fixed *known* f satisfying $\text{supp } f \cap \overline{\Omega} = \emptyset$ and $T < \infty$.

Note that u in $(\mathbf{R}^3 \setminus \overline{\Omega}) \times]0, T[$ can be computed from u on $\partial\Omega \times]0, T[$ by the exactly same formula as (1.2) and thus the problem can be reformulated again as

Inverse Problem II'. Extract information about the location and shape of D from u in $(\mathbf{R}^3 \setminus \overline{\Omega}) \times]0, T[$ for some known f satisfying $\text{supp } f \cap \overline{\Omega} = \emptyset$ and $T < \infty$.

Now we state our second result.

Theorem 1.2. Assume that γ satisfies (A1) or (A2). Let f satisfy (I1) and (I2) and v be the weak solution of (1.4). Let T satisfies (1.8) and w be given by (1.7) with solution of (1.10). If (A1) is satisfied, then there exists a $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS > 0$$

and the formula

$$\lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \log \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS = -\text{dist}(D, B),$$

is valid; if (A2) is satisfied, then there exists a $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS < 0$$

and the formula

$$\lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \log \left(- \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS \right) = -\text{dist}(D, B),$$

is valid.

Isakov [7] considered an inverse problem for the equation $\partial_t^2 u - \nabla \cdot \gamma \nabla u = 0$ in $\Omega \times]-\infty, T[$ with the zero initial data $u = 0$ when $t < 0$ and the lateral Neumann data

$\partial u / \partial \nu = h$ on $\partial \Omega \times] - \infty, T[$, where Ω is the half-space $x_3 < 0$; γ takes the value 1 for $x \in \Omega \setminus D$ and a positive constant k for $x \in D$. The D is given by a Lipschitz continuous function d on \mathbf{R}^2 with $d < 0$ as $D = \{x \mid x_3 < d(x_1, x_2)\}$.

His problem is to recover ∂D by measuring u on $\Gamma \times]0, T[$ for an arbitrary fixed nonempty open set $\Gamma \subset \partial \Omega$ and known h . Choosing h as the lateral Neumann data of a special solution of the wave equation in the half-space, he showed that if $k < 1$, then the data u on $\Gamma \times]0, T[$ uniquely determines the part of D within the set of all points $x = (x_1, x_2, x_3)$ with $x_3 > -T/2$ and $(x_1, x_2) \in \Gamma$. The condition $k < 1$ corresponds to (A1). It is an open problem whether or not the same conclusion holds in the case $k > 1$ which corresponds to (A2). See also [8] for this point.

Rakesh [12] considered an inverse problem for the equation $\gamma \partial_t^2 u - \nabla \cdot (\gamma \nabla u) = 0$ in $\mathbf{R}^3 \times] - \infty, T[$ with $u(x, t) = (t - x_3)_+^2$ for $t \ll 0$. Here $(s)_+^2 = s^2$ if $s > 0$ and $(s)_+^2 = 0$ if $s \leq 0$. The γ takes 1 outside a bounded domain D with smooth boundary and a positive constant $k (\neq 1)$ on D . Thus the governing equation has a same constant speed inside and outside D . The data in his problem is the values of u on $\partial \Omega \times] - \infty, T[$, where Ω is a bounded open set of \mathbf{R}^3 with smooth boundary and satisfies $\overline{D} \subset \Omega$. He showed that; if $T > 6 \text{diam}(\Omega) + \inf_{x \in D} x_3$ and D is *strictly convex*, then the data $\partial u / \partial \nu$ on $\partial \Omega \times] - \infty, T[$ uniquely determine D itself.

Isakov employs a contradiction argument and his method starts with the uniqueness of the continuation of the solution of the wave equation and derives an orthogonality relation that was deduced by denying the conclusion.

Rakesh's argument is also a contradiction argument and makes use of the uniqueness of the continuation of the solution of the wave equation. However, the main point is an analysis of the wave front set of u . See also [13] for other results.

Unlike them we do not make use of the continuation of a wave field nor propagation of singularities argument. The method can be considered as an application of the *enclosure method* which was originally introduced for elliptic equations in [4, 3]. Recently in [5] the author found its application to inverse initial boundary value problems in one-space dimensional case for the heat and wave equations. In [6] we extended this method to the heat equation in two and three-space dimensional cases. Therein the initial data is zero and a *special heat flux* depending on a large parameter is used.

1.3 Further remarks and construction of the paper

Finally we comment on some results in the context of the Lax-Phillips scattering theory. Lax-Phillips in [9] established a relation between the support function of an obstacle and the right end point of the *support* of the *scattering kernel* which is the observation data in their theory. Since the support function gives the signed distance from the origin of coordinates to the support plane of the obstacle, the result means that one can get an estimation of the *convex hull* from the data. Note that the scattering kernel is written by using the scattered wave over the *infinite* time interval that is produced by a singular plane wave at $t \ll 0$ far a way from the obstacle, and thus the data is completely different from ours.

Majda[10] considered the singularity of the scattering kernel and clarified a relation between the support function and the right end point of the *singular support* of the *back scattering kernel*. In [11] a similar result for an obstacle with a finite refractive index is

given. The governing equation has the form $\alpha(x)\partial_t^2 u - \Delta u = 0$ and α has a discontinuity across the boundary of the obstacle and takes 1 outside the obstacle. For other results including the Maxwell equations, hyperbolic systems, etc. we refer the reader to [10, 11] and references therein.

A brief outline of this paper is as follows. Theorems 1.1 and 1.2 are proved in Subsections 2.2 and 3.2, respectively. The key point of the proofs is to derive a lower estimate of the integral

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS, \quad (1.11)$$

where v is the weak solution of (1.4). To establish the estimate we require some integral identities; these identities are found in Subsections 2.1 and 3.1. Using the identities, we show that, if T satisfies (1.8), then the dominant part in the lower estimate of (1.11) in Theorem 1.1 is essentially given by the integral of the square of v over D . We show that this last integral is comparable with $e^{-2\tau \text{dist}(D,B)}$ ignoring a multiplication of a power of τ . This is stated in Subsection 2.2 and proved in Subsection 4.1. Note that in the proof of Theorem 1.2 instead of v the integral of $|\nabla v|^2$ over D plays the same role and the corresponding estimate is stated in Subsection 3.2 and proved in Subsection 4.2. In the final section we give a conclusion of this paper and comments on further problems.

2 The enclosure method for sound hard obstacles

First we specify what we mean by the solution of (1.1). We follow the notion of the weak solution described on pp. 552-566 in [1] and use the notation therein.

By Theorem 1 on p.558 in [1], given $u^0 \in H^1(\mathbf{R}^3 \setminus \overline{D})$ and $u^1 \in L^2(\mathbf{R}^3 \setminus \overline{D})$ we know that there exists a unique u satisfying

$$u \in L^2(0, T; H^1(\mathbf{R}^3 \setminus \overline{D})), u' \in L^2(0, T; H^1(\mathbf{R}^3 \setminus \overline{D})), u'' \in L^2(0, T; (H^1(\mathbf{R}^3 \setminus \overline{D}))')$$

such that, for all $\phi \in H^1(\mathbf{R}^3 \setminus \overline{D})$

$$\langle u''(t), \phi \rangle + \int_{\mathbf{R}^3 \setminus \overline{D}} \nabla u(x, t) \cdot \nabla \phi(x) dx = 0 \text{ a.e. } t \in]0, T[$$

and $u(x, 0) = u^0$, $u'(x, 0) = u^1$. In this section we say that this u for $u^0 = 0$ and $u^1 = f$ is the solution of (1.1).

2.1 A basic identity

Let u be the solution of (1.1). Define

$$w(x; \tau) = \int_0^T e^{-\tau t} u(x, t) dt, \quad x \in \mathbf{R}^3 \setminus \overline{D}.$$

This w belongs to $H^1(\mathbf{R}^3 \setminus \overline{D})$.

Using integration by parts formula (Proposition 2 on p.558 in [DL]), we see that, for all $\phi \in H^1(\mathbf{R}^3 \setminus \overline{D})$ the w satisfies the equation

$$\begin{aligned} & \int_{\mathbf{R}^3 \setminus \overline{D}} \nabla w \cdot \nabla \phi dx + \int_{\mathbf{R}^3 \setminus \overline{D}} (\tau^2 w - f) \phi dx \\ &= -e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (u'(x, T) + \tau u(x, T)) \phi dx. \end{aligned} \quad (2.1)$$

This means that, in a weak sense w satisfies

$$(\Delta - \tau^2)w + f(x) = e^{-\tau T}(u'(x, T) + \tau u(x, T)) \text{ in } \mathbf{R}^3 \setminus \overline{D},$$

$$\frac{\partial w}{\partial \nu} = 0 \text{ on } \partial D.$$

From (2.1) for $v \in C_0^\infty(\mathbf{R}^3 \setminus \overline{D})$, we have: $(\Delta - \tau^2)w + f(x) = e^{-\tau T}(u'(x, T) + \tau u(x, T))$ in $\mathbf{R}^3 \setminus \overline{D}$ in the sense of distribution and hence $\Delta w \in L^2(\mathbf{R}^3 \setminus \overline{D})$. This yields that $w \in H_{\text{loc}}^2(\mathbf{R}^3 \setminus \overline{D})$ and $(\Delta - \tau^2)w + f(x) = e^{-\tau T}(u'(x, T) + \tau u(x, T))$ a.e. $x \in \mathbf{R}^3 \setminus \overline{D}$.

Now define $\partial w / \partial \nu|_{\partial \Omega}$ as $\nabla w|_{\partial \Omega} \cdot \nu \in H^{1/2}(\partial \Omega)$, where $\nabla w|_{\partial \Omega}$ is the trace of ∇w onto $\partial \Omega$.

Let $v \in H^1(\mathbf{R}^3)$ be the weak solution of (1.4). For this v by the same reason as above we have $v \in H_{\text{loc}}^2(\mathbf{R}^3)$ and thus $\partial v / \partial \nu|_{\partial \Omega} \equiv \nabla v|_{\partial \Omega} \cdot \nu \in H^{1/2}(\partial \Omega)$.

In this subsection we derive the following identity.

Proposition 2.1. *Let $v \in H^1(\mathbf{R}^3)$ be the weak solution of (1.4). It holds that*

$$\begin{aligned} & \int_{\partial \Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS \\ &= \int_D |\nabla v|^2 dx + \tau^2 \int_D |v|^2 dx + \int_{\mathbf{R}^3 \setminus \overline{D}} |\nabla(w - v)|^2 dx + \tau^2 \int_{\mathbf{R}^3 \setminus \overline{D}} |w - v|^2 dx \\ &+ e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (w - v)(u'(x, T) + \tau u(x, T)) dx - e^{-\tau T} \int_{\Omega \setminus \overline{D}} (u'(x, T) + \tau u(x, T)) v dx \\ &- \int_{\Omega \setminus \overline{D}} f(w - v) dx - \int_D f v dx. \end{aligned} \quad (2.2)$$

Proof. First we prove that

$$\begin{aligned} & \int_{\partial \Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS \\ &= \int_{\partial D} \frac{\partial v}{\partial \nu} w - \int_{\Omega \setminus \overline{D}} f(w - v) dx - e^{-\tau T} \int_{\Omega \setminus \overline{D}} (u'(x, T) + \tau u(x, T)) v(x) dx. \end{aligned} \quad (2.3)$$

Let $\varphi \in H^1(\Omega \setminus \overline{D})$ satisfy $\varphi = 0$ on $\partial \Omega$ in the sense of the trace. Since the zero extension of this φ belongs to $H^1(\mathbf{R}^3 \setminus \overline{D})$, it follows from (2.1) that

$$\int_{\Omega \setminus \overline{D}} \nabla w \cdot \nabla \varphi + \int_{\Omega \setminus \overline{D}} \{ \tau^2 w - f + e^{-\tau T}(u'(x, T) + \tau u(x, T)) \} \varphi dx = 0. \quad (2.4)$$

Choose $\chi \in C_0^\infty(\mathbf{R}^3)$ such that $\chi(x) \equiv 1$ in a neighbourhood of $\partial\Omega$ and $\chi(x) \equiv 0$ in a neighbourhood of \overline{D} . Since $(1 - \chi)v|_{\Omega \setminus \overline{D}}$ vanishes in a neighbourhood of $\partial\Omega$, it follows that (2.4) is valid for $\varphi = (1 - \chi)v|_{\Omega \setminus \overline{D}}$.

On the other hand, since χv vanishes in a neighbourhood of \overline{D} and $w \in H_{\text{loc}}^2(\mathbf{R}^3 \setminus \overline{D})$, integration by parts yields

$$\begin{aligned} & \int_{\Omega \setminus \overline{D}} \nabla w \cdot \nabla(\chi v) dx \\ &= \int_{\partial\Omega} \frac{\partial w}{\partial \nu} \chi v dS - \int_{\Omega \setminus \overline{D}} (\Delta w) \chi v dx \\ &= \int_{\partial\Omega} \frac{\partial w}{\partial \nu} v dS - \int_{\Omega \setminus \overline{D}} \{\tau^2 w - f + e^{-\tau T}(u'(x, T) + \tau u(x, T))\} \chi v dx. \end{aligned}$$

From this and (2.4) for $\varphi = (1 - \chi)v|_{\Omega \setminus \overline{D}}$ we obtain

$$\int_{\partial\Omega} \frac{\partial w}{\partial \nu} v dS = \int_{\Omega \setminus \overline{D}} \nabla w \cdot \nabla v dx + \int_{\Omega \setminus \overline{D}} \{\tau^2 w - f + e^{-\tau T}(u'(x, T) + \tau u(x, T))\} v dx. \quad (2.5)$$

Choose $\tilde{w} \in H^1(\Omega)$ such that $\tilde{w} = w$ in $\Omega \setminus \overline{D}$. Note that $\tilde{w} = w$ on $\partial\Omega$ and ∂D in the sense of the trace. Since $v \in H^2(\Omega)$, we have

$$\begin{aligned} & \int_{\partial\Omega} \frac{\partial v}{\partial \nu} w dS = \int_{\partial\Omega} \frac{\partial v}{\partial \nu} \tilde{w} dS \\ &= \int_{\Omega} \Delta v \tilde{w} dx + \int_{\Omega} \nabla v \cdot \nabla \tilde{w} dx \\ &= \int_{\Omega \setminus \overline{D}} (\tau^2 v - f) w dx + \int_{\Omega \setminus \overline{D}} \nabla v \cdot \nabla w dx \\ &+ \int_D (\tau^2 v - f) \tilde{w} dx + \int_D \nabla v \cdot \nabla \tilde{w} dx. \end{aligned}$$

On the other hand we have

$$\begin{aligned} & \int_D \nabla v \cdot \nabla \tilde{w} dx = \int_{\partial D} \frac{\partial v}{\partial \nu} \tilde{w} - \int_D (\Delta v) \tilde{w} dx \\ &= \int_{\partial D} \frac{\partial v}{\partial \nu} w dS - \int_D (\tau^2 v - f) \tilde{w} dx, \end{aligned}$$

that is

$$\int_{\partial D} \frac{\partial v}{\partial \nu} w dS = \int_D (\tau^2 v - f) \tilde{w} dx + \int_D \nabla v \cdot \nabla \tilde{w} dx.$$

Therefore we obtain

$$\int_{\partial\Omega} \frac{\partial v}{\partial \nu} w dS = \int_{\Omega \setminus \overline{D}} (\tau^2 v - f) w dx + \int_{\Omega \setminus \overline{D}} \nabla v \cdot \nabla w dx + \int_{\partial D} \frac{\partial v}{\partial \nu} w dS. \quad (2.6)$$

A combination of (2.5) and (2.6) gives (2.3).

Write

$$\int_{\partial D} \frac{\partial v}{\partial \nu} w dS = \int_{\partial D} \frac{\partial v}{\partial \nu} (w - v) dS + \int_{\partial D} \frac{\partial v}{\partial \nu} v dS. \quad (2.7)$$

It follows from (1.5) and the trace theorem that, for all $\phi \in H^1(\mathbf{R}^3 \setminus \overline{D})$

$$\int_{\partial D} \frac{\partial v}{\partial \nu} \phi dS + \int_{\mathbf{R}^3 \setminus \overline{D}} \nabla v \cdot \nabla \phi dx + \tau^2 \int_{\mathbf{R}^3 \setminus \overline{D}} v \phi dx = \int_{\mathbf{R}^3 \setminus \overline{D}} f \phi dx.$$

Combining this with (2.1), we obtain

$$\begin{aligned} & \int_{\partial D} \frac{\partial v}{\partial \nu} \phi dS - \int_{\mathbf{R}^3 \setminus \overline{D}} \nabla(w - v) \cdot \nabla \phi dx - \tau^2 \int_{\mathbf{R}^3 \setminus \overline{D}} (w - v) \phi dx \\ &= e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (u'(x, T) + \tau u(x, T)) \phi dx. \end{aligned} \quad (2.8)$$

This means that $w - v$ satisfies, in a weak sense

$$(\Delta - \tau^2)(w - v) = e^{-\tau T} (u'(x, T) + \tau u(x, T)) \text{ in } \mathbf{R}^3 \setminus \overline{D},$$

$$\frac{\partial}{\partial \nu} (w - v) = -\frac{\partial v}{\partial \nu} \text{ on } \partial D.$$

Substituting $w - v$ for ϕ in (2.8), we obtain

$$\begin{aligned} \int_{\partial D} \frac{\partial v}{\partial \nu} (w - v) dS &= \int_{\mathbf{R}^3 \setminus \overline{D}} |\nabla(w - v)|^2 dx + \tau^2 \int_{\mathbf{R}^3 \setminus \overline{D}} |w - v|^2 dx \\ &+ e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (u'(x, T) + \tau u(x, T)) (w - v). \end{aligned} \quad (2.9)$$

Now from this together with (2.3), (2.7) and the identity

$$\int_{\partial D} \frac{\partial v}{\partial \nu} v dS = \tau^2 \int_D |v|^2 dx + \int_D |\nabla v|^2 dx - \int_D f v dx, \quad (2.10)$$

we obtain (2.2).

□

In particular, choose f in such a way that $\text{supp } f \cap \overline{\Omega} = \emptyset$. Then (2.2) becomes

$$\begin{aligned} & \int_{\partial \Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS \\ &= \int_D |\nabla v|^2 dx + \tau^2 \int_D |v|^2 dx + \int_{\mathbf{R}^3 \setminus \overline{D}} |\nabla(w - v)|^2 dx + \tau^2 \int_{\mathbf{R}^3 \setminus \overline{D}} |w - v|^2 dx \\ &+ e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (w - v) (u'(x, T) + \tau u(x, T)) dx - e^{-\tau T} \int_{\Omega \setminus \overline{D}} (u'(x, T) + \tau u(x, T)) v dx. \end{aligned} \quad (2.11)$$

This is the basic identity for the sound-hard obstacles.

2.2 Proof of Theorem 1.1.

Using the identity

$$\begin{aligned} & \tau^2 |w - v|^2 + e^{-\tau t} (w - v) (u'(x, T) + \tau u(x, T)) \\ &= \left| \tau(w - v) + \frac{e^{-\tau T}}{2\tau} (u'(x, T) + \tau u(x, T)) \right|^2 - \frac{e^{-2\tau T}}{4\tau^2} |u'(x, T) + \tau u(x, T)|^2, \end{aligned}$$

we have from (2.11)

$$\begin{aligned} & \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS \geq \tau^2 \int_D |v|^2 dx \\ & - \frac{e^{-2\tau T}}{4\tau^2} \int_{\mathbf{R}^3 \setminus \overline{D}} |u'(x, T) + \tau u(x, T)|^2 dx - e^{-\tau T} \int_{\Omega \setminus \overline{D}} (u'(x, T) + \tau u(x, T)) v dx. \end{aligned} \quad (2.12)$$

From (1.6) we have, for all $x \in \mathbf{R}^3 \setminus \overline{B}$

$$|v(x)| \leq \frac{e^{-\tau d_B(x)}}{4\pi d_B(x)} \|f\|_{L^1(B)}, \quad |\nabla v(x)| \leq \frac{e^{-\tau d_B(x)}}{4\pi} \left(\tau + \frac{1}{d_B(x)^2} \right) \|f\|_{L^1(B)}. \quad (2.13)$$

This gives $\|v\|_{L^2(\Omega \setminus \overline{D})} = O(e^{-\tau \text{dist}(\Omega, B)})$ and note that

$$\int_{\mathbf{R}^3 \setminus \overline{D}} |u'(x, T) + \tau u(x, T)|^2 dx = O(\tau^2).$$

Using these estimates, we obtain

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS \geq \tau^2 \int_D |v|^2 dx + O(e^{-2\tau T}) + O(\tau e^{-\tau T} e^{-\tau \text{dist}(\Omega, B)}). \quad (2.14)$$

Here we state a key lemma whose proof is given in Section 4.

Lemma 2.1. *It holds that*

$$\liminf_{\tau \rightarrow \infty} \tau^6 e^{2\tau \text{dist}(D, B)} \int_D |v|^2 dx > 0. \quad (2.15)$$

Multiplying the both side of (2.14) by $\tau^4 e^{2\tau \text{dist}(D, B)}$, we have

$$\begin{aligned} & \tau^4 e^{2\tau \text{dist}(D, B)} \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS \geq \tau^6 e^{2\tau \text{dist}(D, B)} \int_D |v|^2 dx \\ & + O(\tau^4 e^{-2\tau(T - \text{dist}(D, B))}) + O(\tau^5 e^{-\tau(T - 2\text{dist}(D, B) + \text{dist}(\Omega, B))}) \end{aligned}$$

and thus from (2.15) one gets

$$\liminf_{\tau \rightarrow \infty} \tau^4 e^{2\tau \text{dist}(D, B)} \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS > 0 \quad (2.16)$$

provided if $T > 2\text{dist}(D, B) - \text{dist}(\Omega, B)$.

On the other hand, using (2.3), (2.7) and (2.10) one has

$$\begin{aligned} \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS &= \int_D |\nabla v|^2 dx + \tau^2 \int_D |v|^2 dx \\ &+ \int_{\partial D} \frac{\partial v}{\partial \nu} (w - v) dS + O(\tau e^{-\tau T} e^{-\tau \text{dist}(\Omega, B)}). \end{aligned} \quad (2.17)$$

From (2.13) we have, as $\tau \rightarrow \infty$

$$\int_D |\nabla v|^2 dx + \tau^2 \int_D |v|^2 dx = O(\tau^2 e^{-2\tau \text{dist}(D, B)}). \quad (2.18)$$

Concerning with the bound on the third term of the right hand side of (2.17), we have the following lemma.

Lemma 2.2. *It holds that, as $\tau \rightarrow \infty$*

$$\left\| \frac{\partial v}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} = O(\tau^2 e^{-\tau \text{dist}(D, B)}) \quad (2.19)$$

and

$$\|w - v\|_{H^1(\mathbf{R}^3 \setminus \overline{D})} = O(\tau e^{-\tau T} + \tau^2 e^{-\tau \text{dist}(D, B)}). \quad (2.20)$$

Proof. Set $\epsilon = w - v$. It follows from (2.9) that

$$\int_{\mathbf{R}^3 \setminus \overline{D}} |\nabla \epsilon|^2 dx + \tau^2 \int_{\mathbf{R}^3 \setminus \overline{D}} |\epsilon|^2 dx = -e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (u'(x, T) + \tau u(x, T)) \epsilon dx - \int_{\partial D} \frac{\partial \epsilon}{\partial \nu} \epsilon dS$$

and thus this yields

$$\begin{aligned} &\|\nabla \epsilon\|_{L^2(\mathbf{R}^3 \setminus \overline{D})}^2 + \tau^2 \|\epsilon\|_{L^2(\mathbf{R}^3 \setminus \overline{D})}^2 \\ &\leq e^{-\tau T} \|u'(\cdot, T) + \tau u(\cdot, T)\|_{L^2(\mathbf{R}^3 \setminus \overline{D})} \|\epsilon\|_{L^2(\mathbf{R}^3 \setminus \overline{D})} + \left| \int_{\partial D} \frac{\partial v}{\partial \nu} \epsilon dS \right| \end{aligned} \quad (2.21)$$

The boundedness of the trace operator $H^1(\mathbf{R}^3 \setminus \overline{D}) \rightarrow H^{1/2}(\partial D)$ yields

$$\begin{aligned} \left| \int_{\partial D} \frac{\partial v}{\partial \nu} \epsilon dS \right| &\leq \left\| \frac{\partial v}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} \|\epsilon|_{\partial D}\|_{H^{1/2}(\partial D)} \\ &\leq C \left\| \frac{\partial v}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} \|\epsilon\|_{H^1(\mathbf{R}^3 \setminus \overline{D})}, \end{aligned} \quad (2.22)$$

where C is a positive constant independent of τ . Now from (2.21) and (2.22) we obtain, for all $\delta > 0$

$$\begin{aligned} &(1 - C\delta^2) \|\nabla \epsilon\|_{L^2(\mathbf{R}^3 \setminus \overline{D})}^2 + \left(\tau^2 - C\delta^2 - \frac{\delta^2}{2} \right) \|\epsilon\|_{L^2(\mathbf{R}^3 \setminus \overline{D})}^2 \\ &\leq \frac{\delta^{-2}}{2} e^{-2\tau T} \|u'(\cdot, T) + \tau u(\cdot, T)\|_{L^2(\mathbf{R}^3 \setminus \overline{D})}^2 + C\delta^{-2} \left\| \frac{\partial v}{\partial \nu} \right\|_{H^{-1/2}(\partial D)}^2. \end{aligned} \quad (2.23)$$

Since v satisfies $(\Delta - \tau^2)v = 0$ in D , for all $\Psi \in H^1(D)$ we have

$$\int_{\partial D} \frac{\partial v}{\partial \nu} \Psi dS = \tau^2 \int_D v \Psi dx + \int_D \nabla v \cdot \nabla \Psi dx.$$

The trace operator $H^1(D) \longrightarrow H^{1/2}(\partial D)$ has a bounded right inverse. This together with the identity above yields

$$\left\| \frac{\partial v}{\partial \nu} \right\|_{H^{-1/2}(\partial D)} \leq C(\|\nabla v\|_{L^2(D)} + \tau^2 \|v\|_{L^2(D)}),$$

where C is a positive constant independent of τ . It follows from (2.13) that this right hand side has the bound $O(\tau^2 e^{-\tau \text{dist}(D,B)})$. Thus (2.19) is valid. Now (2.20) is a consequence of (2.19) and (2.23).

□

Now we continue the proof of Theorem 1.1.

It follows from (2.19), (2.20) and (2.22) that

$$\int_{\partial D} \frac{\partial v}{\partial \nu} (w - v) dS = O(\tau^3 e^{-\tau(T + \text{dist}(D,B))} + \tau^4 e^{-2\tau \text{dist}(D,B)}).$$

From this, (2.17), (2.18) we obtain

$$\begin{aligned} & e^{2\tau \text{dist}(D,B)} \int_{\partial \Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS \\ &= O(\tau^3 e^{-\tau(T - \text{dist}(D,B))} + \tau^4 + \tau e^{-\tau(T - 2\text{dist}(D,B) + \text{dist}(\Omega, B))}). \end{aligned}$$

A combination of this and the estimate $\text{dist}(D, B) > \text{dist}(\Omega, B)$ gives

$$\limsup_{\tau \rightarrow \infty} \tau^{-4} e^{2\tau \text{dist}(D,B)} \left| \int_{\partial \Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS \right| < \infty \quad (2.24)$$

provided $T > 2 \text{dist}(D, B) - \text{dist}(\Omega, B)$. Now the formula (1.9) is a direct consequence of (2.16) and (2.24).

□

3 The enclosure method for penetrable obstacles

First we specify what we mean by the solution of (1.10). By Theorem 1 on p.558 in [1], given $u^0 \in H^1(\mathbf{R}^3)$ and $u^1 \in L^2(\mathbf{R}^3)$ we know that there exists a unique u satisfying

$$u \in L^2(0, T; H^1(\mathbf{R}^3)), u' \in L^2(0, T; H^1(\mathbf{R}^3)), u'' \in L^2(0, T; (H^1(\mathbf{R}^3))')$$

such that, for all $\phi \in H^1(\mathbf{R}^3)$

$$\langle u''(t), \phi \rangle + \int_{\mathbf{R}^3} \gamma(x) \nabla u(x, t) \cdot \nabla \phi(x) dx = 0 \text{ a.e. } t \in]0, T[$$

and $u(x, 0) = u^0$, $u'(x, 0) = u^1$. In this section we say that this u for $u^0 = 0$ and $u^1 = f$ is the solution of (1.10).

3.1 A basic identity

Let u be the solution of (1.10). Define

$$w(x; \tau) = \int_0^T e^{-\tau t} u(x, t) dt, \quad x \in \mathbf{R}^3.$$

This w belongs to $H^1(\mathbf{R}^3)$.

From integration by parts (Proposition 2 on p.558 in [1]) it follows that, for all $\phi \in H^1(\mathbf{R}^3)$

$$\int_{\mathbf{R}^3} \gamma \nabla w \cdot \nabla \phi dx + \int_{\mathbf{R}^3} (\tau^2 w - f) \phi dx = -e^{-\tau T} \int_{\mathbf{R}^3} (u'(x, T) + \tau u(x, T)) \phi dx. \quad (3.1)$$

This means that in a weak sense w satisfies

$$(\nabla \cdot \gamma \nabla - \tau^2)w + f(x) = e^{-\tau T} (u'(x, T) + \tau u(x, T)) \text{ in } \mathbf{R}^3.$$

By a similar reason as the sound-hard obstacle case we know that $w \in H_{\text{loc}}^2(\mathbf{R}^3 \setminus \overline{D})$. Since $\gamma(x) \equiv 1$ in $\mathbf{R}^3 \setminus \overline{D}$, we define $\gamma \nabla w \cdot \nu|_{\partial\Omega}$ as $\nabla w|_{\partial\Omega} \cdot \nu$, where $\nabla w|_{\partial\Omega}$ denotes the trace of ∇w onto $\partial\Omega$. Note also that w satisfies $(\Delta - \tau^2)w + f(x) = e^{-\tau T} (u'(x, T) + \tau u(x, T))$ a.e. $x \in \mathbf{R}^3$.

In this subsection we derive an important identity.

Proposition 3.1. *Let v be the weak solution of (1.4). It holds that*

$$\begin{aligned} & \int_{\partial\Omega} \{(\nabla v \cdot \nu)w - (\gamma \nabla w \cdot \nu)v\} dS \\ &= - \int_D h \nabla v \cdot \nabla v dx + \int_{\mathbf{R}^3} \gamma \nabla(w - v) \cdot \nabla(w - v) dx + \tau^2 \int_{\mathbf{R}^3} |w - v|^2 dx - \int_{\Omega} f(w - v) dx \\ & \quad + e^{-\tau T} \int_{\mathbf{R}^3} (u'(x, T) + \tau u(x, T))(w - v) dx - e^{-\tau T} \int_{\Omega} (u'(x, T) + \tau u(x, T))v dx. \end{aligned} \quad (3.2)$$

Proof. Using a similar argument for the proof of (2.5) and (2.6), we obtain

$$\int_{\partial\Omega} (\gamma \nabla w \cdot \nu) v dS = \int_{\Omega} \gamma \nabla w \cdot \nabla v dx + \int_{\Omega} \{\tau^2 w - f + e^{-\tau T} (u'(x, T) + \tau u(x, T))\} v dx$$

and

$$\int_{\partial\Omega} (\nabla v \cdot \nu) w dS = \int_{\Omega} \nabla v \cdot \nabla w dx + \int_{\Omega} (\tau^2 v - f) w dx.$$

From these we obtain

$$\begin{aligned} & \int_{\partial\Omega} \{(\nabla v \cdot \nu)w - (\gamma \nabla w \cdot \nu)v\} dS \\ &= - \int_D h \nabla w \cdot \nabla v dx - \int_{\Omega} f(w - v) dx - e^{-\tau T} \int_{\Omega} (u'(x, T) + \tau u(x, T))v dx. \end{aligned} \quad (3.3)$$

Write

$$- \int_D h \nabla w \cdot \nabla v dx = - \int_D h \nabla v \cdot \nabla v dx - \int_D h \nabla(w - v) \cdot \nabla v dx. \quad (3.4)$$

Since v satisfies (1.5), from (3.1) we have, for all $\phi \in H^1(\mathbf{R}^3)$

$$\begin{aligned} & - \int_{\mathbf{R}^3} \gamma \nabla(w-v) \cdot \nabla \phi dx - \tau^2 \int_{\mathbf{R}^3} (w-v) \phi dx \\ & = - \int_{\mathbf{R}^3} (I_3 - \gamma) \nabla v \cdot \nabla \phi dx + e^{-\tau T} \int_{\mathbf{R}^3} (u'(x, T) + \tau u(x, T)) \phi dx. \end{aligned} \quad (3.5)$$

This means that the $w-v$ satisfies, in a weak sense

$$(\nabla \cdot \gamma \nabla - \tau^2)(w-v) = \nabla \cdot (I_3 - \gamma) \nabla v + e^{-\tau T} (u'(x, T) + \tau u(x, T)) \text{ in } \mathbf{R}^3.$$

Substituting $w-v$ for ϕ in (3.5), we obtain

$$\begin{aligned} & \int_{\mathbf{R}^3} (I_3 - \gamma) \nabla v \cdot \nabla (w-v) dx = \int_{\mathbf{R}^3} \gamma \nabla(w-v) \cdot \nabla(w-v) dx \\ & + \tau^2 \int_{\mathbf{R}^3} (w-v)(w-v) dx + e^{-\tau T} \int_{\mathbf{R}^3} (u'(x, T) + \tau u(x, T))(w-v) dx. \end{aligned} \quad (3.6)$$

A combination of (3.4) and (3.6) gives

$$\begin{aligned} & - \int_D h \nabla w \cdot \nabla v dx = - \int_D h \nabla v \cdot \nabla v dx \\ & + \int_{\mathbf{R}^3} \gamma \nabla(w-v) \cdot \nabla(w-v) dx + \tau^2 \int_{\mathbf{R}^3} (w-v)(w-v) dx \\ & + e^{-\tau T} \int_{\mathbf{R}^3} (u'(x, T) + \tau u(x, T))(w-v) dx. \end{aligned}$$

Now from this and (3.3) we obtain (3.2).

□

In particular, choose f in such a way that $\text{supp } f \cap \overline{\Omega} = \emptyset$. Then (3.2) become

$$\begin{aligned} & \int_{\partial\Omega} \{(\nabla v \cdot \nu)w - (\gamma \nabla w \cdot \nu)v\} dS \\ & = - \int_D h \nabla v \cdot \nabla v dx + \int_{\mathbf{R}^3} \gamma \nabla(w-v) \cdot \nabla(w-v) dx + \tau^2 \int_{\mathbf{R}^3} |w-v|^2 dx \\ & + e^{-\tau T} \int_{\mathbf{R}^3} (u'(x, T) + \tau u(x, T))(w-v) dx - e^{-\tau T} \int_{\Omega} (u'(x, T) + \tau u(x, T))v dx. \end{aligned} \quad (3.7)$$

This is our first basic identity which is useful in the proof of Theorem 1.2 under the assumption (A.1).

Unfortunately, for (A2) this identity does not work. However, one can rewrite this by replacing the role of v and w in the proof of Proposition 3.1. More precisely, set $\tilde{f} = f - e^{-\tau T} (u'(x, T) + \tau u(x, T))$. The points are: w satisfies $\nabla \cdot \gamma \nabla w - \tau^2 w + \tilde{f} = 0$ in \mathbf{R}^3 and v satisfies $\nabla \cdot I_3 \nabla v - \tau^2 v + \tilde{f} = -e^{-\tau T} (u'(x, T) + \tau u(x, T))$ in \mathbf{R}^3 . Thus changing the role of v and w in the proof of Proposition 3.1, we can easily obtain another expression of (3.2).

Proposition 3.2. *Let v be the weak solution of (1.4). It holds that*

$$\begin{aligned}
& \int_{\partial\Omega} \{(\gamma \nabla w \cdot \nu)v - (\nabla v \cdot \nu)w\} dS \\
&= \int_D h \nabla w \cdot \nabla w dx + \int_{\mathbf{R}^3} \nabla(v-w) \cdot \nabla(v-w) dx + \tau^2 \int_{\mathbf{R}^3} |v-w|^2 dx - \int_{\Omega} f(v-w) dx \\
& \quad - e^{-\tau T} \int_{\mathbf{R}^3} (u'(x, T) + \tau u(x, T))(v-w) dx + e^{-\tau T} \int_{\Omega} (u'(x, T) + \tau u(x, T))v dx.
\end{aligned} \tag{3.8}$$

In particular, if $\text{supp } f \cap \overline{\Omega} = \emptyset$, then (3.8) gives

$$\begin{aligned}
& \int_{\partial\Omega} \{(\gamma \nabla w \cdot \nu)v - (\nabla v \cdot \nu)w\} dS \\
&= \int_D h \nabla w \cdot \nabla w dx + \int_{\mathbf{R}^3} \nabla(v-w) \cdot \nabla(v-w) dx + \tau^2 \int_{\mathbf{R}^3} |v-w|^2 dx \\
& \quad - e^{-\tau T} \int_{\mathbf{R}^3} (u'(x, T) + \tau u(x, T))(v-w) dx + e^{-\tau T} \int_{\Omega} (u'(x, T) + \tau u(x, T))v dx.
\end{aligned} \tag{3.9}$$

3.2 Proof of Theorem 1.2.

First we consider the case when (A1) is satisfied. Using (A1) and the identity

$$\begin{aligned}
& \tau^2 |w-v|^2 + e^{-\tau t} (w-v)(u'(x, T) + \tau u(x, T)) \\
&= \left| \tau(w-v) + \frac{e^{-\tau T}}{2\tau} (u'(x, T) + \tau u(x, T)) \right|^2 - \frac{e^{-2\tau T}}{4\tau^2} |u'(x, T) + \tau u(x, T)|^2,
\end{aligned}$$

we have from (3.7)

$$\begin{aligned}
& \int_{\partial\Omega} \{(\nabla v \cdot \nu)w - (\gamma \nabla w \cdot \nu)v\} dS \geq C \int_D |\nabla v|^2 dx \\
& \quad - \frac{e^{-2\tau T}}{4\tau^2} \int_{\mathbf{R}^3} |u'(x, T) + \tau u(x, T)|^2 dx - e^{-\tau T} \int_{\Omega} (u'(x, T) + \tau u(x, T))v dx.
\end{aligned} \tag{3.10}$$

Since we have

$$\int_{\mathbf{R}^3} |u'(x, T) + \tau u(x, T)|^2 dx = O(\tau^2)$$

and the estimate $\|v\|_{L^2(\Omega)} = O(e^{-\tau \text{dist}(\Omega, B)})$, it follows from (3.10) that

$$\int_{\partial\Omega} \{(\nabla v \cdot \nu)w - (\gamma \nabla w \cdot \nu)v\} dS \geq C \int_D |\nabla v|^2 dx + O(e^{-2\tau T}) + O(\tau e^{-\tau T} e^{-\tau \text{dist}(\Omega, B)}). \tag{3.11}$$

Here we state a key lemma whose proof is given in the next section.

Lemma 3.1. *It holds that*

$$\liminf_{\tau \rightarrow \infty} \tau^4 e^{2\tau} \text{dist}(D, B) \int_D |\nabla v|^2 dx > 0. \tag{3.12}$$

Multiplying the both side of (3.11) by $\tau^\mu e^{2\tau \text{dist}(D,B)}$, we have

$$\begin{aligned} \tau^4 e^{2\tau \text{dist}(D,B)} \int_{\partial\Omega} \{(\nabla v \cdot \nu)w - (\gamma \nabla w \cdot \nu)v\} dS &\geq \tau^4 e^{2\tau \text{dist}(D,B)} \int_D |\nabla v|^2 dx \\ &+ O(\tau^4 e^{-2\tau(T-\text{dist}(D,B))}) + O(\tau^5 e^{-\tau(T-2\text{dist}(D,B)+\text{dist}(\Omega,B))}) \end{aligned}$$

and thus from (3.12) one gets

$$\liminf_{\tau \rightarrow \infty} \tau^4 e^{2\tau \text{dist}(D,B)} \int_{\partial\Omega} \{(\nabla v \cdot \nu)w - (\gamma \nabla w \cdot \nu)v\} dS > 0 \quad (3.13)$$

provided if $T > 2\text{dist}(D, B) - \text{dist}(\Omega, B)$.

On the other hand, using (3.3) and (3.4), one gets

$$\begin{aligned} &\int_{\partial\Omega} \{(\nabla v \cdot \nu)w - (\gamma \nabla w \cdot \nu)v\} dS \\ &= - \int_D h \nabla v \cdot \nabla v dx - \int_D h \nabla(w-v) \cdot \nabla v dx + O(\tau e^{-\tau T} e^{-\tau \text{dist}(\Omega, B)}). \end{aligned} \quad (3.14)$$

From (2.13) we have, as $\tau \rightarrow \infty$

$$\int_D |\nabla v|^2 dx = O(\tau^2 e^{-2\tau \text{dist}(D,B)}). \quad (3.15)$$

Concerning with the bound on the second term of the right hand side of (3.12), we have the following lemma.

Lemma 3.2. *It holds that, as $\tau \rightarrow \infty$*

$$\|w - v\|_{H^1(\mathbf{R}^3)} = O(\tau e^{-\tau T} + \tau e^{-\tau \text{dist}(D,B)}) \quad (3.16)$$

Proof. Set $\epsilon = w - v$. From (3.6) we have

$$- \int_D h \nabla v \cdot \nabla \epsilon dx - \int_{\mathbf{R}^3} \gamma \nabla \epsilon \cdot \nabla \epsilon dx - \tau^2 \int_{\mathbf{R}^3} |\epsilon|^2 dx = e^{-\tau T} \int_{\mathbf{R}^3} (u'(x, T) + \tau u(x, T)) \epsilon dx.$$

This yields

$$\begin{aligned} &C \|\nabla \epsilon\|_{L^2(\mathbf{R}^3)}^2 + \tau^2 \|\epsilon\|_{L^2(\mathbf{R}^3)}^2 \\ &\leq e^{-\tau T} \|u'(\cdot, T) + \tau u(\cdot, T)\|_{L^2(\mathbf{R}^3)} \|\epsilon\|_{L^2(\mathbf{R}^3)} + \left| \int_D h \nabla v \cdot \nabla \epsilon dx \right| \end{aligned}$$

and thus one gets, for all $\delta > 0$

$$\begin{aligned} &\left(C - \frac{\delta^2}{2}\right) \|\nabla \epsilon\|_{L^2(\mathbf{R}^3)}^2 + \left(\tau^2 - \frac{\delta^2}{2}\right) \|\epsilon\|_{L^2(\mathbf{R}^3)}^2 \\ &\leq \frac{\delta^{-2}}{2} e^{-2\tau T} \|u'(\cdot, T) + \tau u(\cdot, T)\|_{L^2(\mathbf{R}^3)}^2 + \frac{\delta^{-2}}{2} \|h\|_{L^\infty(D)}^2 \|\nabla v\|_{L^2(D)}^2. \end{aligned}$$

Now a combination of this and (3.15) yields (3.16).

□

A combination of (3.15) and (3.16) gives

$$\int_D h \nabla(w - v) \cdot \nabla v dx = O(\tau^2 e^{-\tau(T + \text{dist}(D, B))} + \tau^2 e^{-2\tau \text{dist}(D, B)}).$$

From this, (3.14), (3.15) we obtain

$$\begin{aligned} & e^{2\tau \text{dist}(D, B)} \int_{\partial\Omega} ((\nabla v \cdot \nu)w - (\gamma \nabla w \cdot \nu)v) dS \\ &= O(\tau^2 e^{-\tau(T - \text{dist}(D, B))} + \tau^2 + \tau e^{-\tau(T - 2\text{dist}(D, B) + \text{dist}(\Omega, B))}). \end{aligned}$$

This together with the estimate $\text{dist}(D, B) > \text{dist}(\Omega, B)$ yields

$$\limsup_{\tau \rightarrow \infty} \tau^{-2} e^{2\tau \text{dist}(D, B)} \left| \int_{\partial\Omega} ((\nabla v \cdot \nu)w - (\gamma \nabla w \cdot \nu)v) dS \right| < \infty \quad (3.17)$$

provided $T > 2 \text{dist}(D, B) - \text{dist}(\Omega, B)$. Now the conclusion of Theorem 1.2 is a direct consequence of (3.13) and (3.17).

□

Finally we give a comment on the case when (A2) is satisfied. In this case we make use of (3.9) instead of (3.7). A combination of the well known inequality (see [2])

$$(\gamma(x) - I_3) \nabla w \cdot \nabla w + \nabla(w - v) \cdot \nabla(w - v) \geq (\gamma(x) - I_3) \gamma(x)^{-1/2} \nabla v \cdot \gamma(x)^{-1/2} \nabla v$$

and (A2) yields that there exists a positive constant C such that

$$h(x) \nabla w \cdot \nabla w + \nabla(w - v) \cdot \nabla(w - v) \geq C |\nabla v|^2$$

for a.e. $x \in D$. This together with (3.9) gives the lower estimate

$$\begin{aligned} & \int_{\partial\Omega} \{(\gamma \nabla w \cdot \nu)v - (\nabla v \cdot \nu)w\} dS \geq C \int_D |\nabla v|^2 dx \\ & - \frac{e^{-2\tau T}}{4\tau^2} \int_{\mathbb{R}^3} |u'(x, T) + \tau u(x, T)|^2 dx + e^{-\tau T} \int_{\Omega} (u'(x, T) + \tau u(x, T)) v dx. \end{aligned}$$

which corresponds to (3.10). Applying the argument for the proof of (3.13) to this right hand side we obtain

$$\liminf_{\tau \rightarrow \infty} \tau^4 e^{2\tau \text{dist}(D, B)} \int_{\partial\Omega} \{(\gamma \nabla w \cdot \nu)v - (\nabla v \cdot \nu)w\} dS > 0.$$

Since (3.17) is valid also for case (A2) we obtain the desired conclusion.

□

4 Proof of Lemmas 2.1 and 3.1

4.1 Proof of Lemma 2.1.

Choose points $x_0 \in \partial D$ and $y_0 \in \partial B$ such that $\text{dist}(D, B) = |x_0 - y_0|$. Since we have assumed that ∂D is smooth, one can find an open ball B' such that $B' \subset D$ and $x_0 \in$

$\partial B' \cap \partial D$. Since $\text{dist}(B', B) = |x_0 - y_0|$, it suffices to prove (2.15) in the case when $D = B'$.

Write

$$v(x)^2 = \left(\frac{1}{4\pi}\right)^2 \int_{B \times B} \frac{e^{-\tau(|y_1-x|+|y_2-x|)}}{|x-y_1||x-y_2|} f(y_1)f(y_2)dy_1dy_2.$$

It follows from the assumption on f that

$$v(x)^2 \geq C^2 I(x, \tau)^2 \quad (4.1)$$

where

$$I(x, \tau) = \frac{1}{4\pi} \int_B \frac{e^{-\tau|y-x|}}{|y-x|} dy, \quad x \in B'.$$

We denote by p and η the center and radius of B , respectively. Using the polar coordinates centered at x , one can write

$$B = \{y = x + r\omega \mid \omega \in S(x, B), r^+(\omega) < r < r^-(\omega)\}$$

where

$$S(x, B) = \{\omega \in S^2 \mid \omega \cdot (p - x) > \sqrt{|x - p|^2 - \eta^2}\}$$

and

$$r^\pm(\omega) = \omega \cdot (p - x) \mp \sqrt{(\omega \cdot (p - x))^2 - |x - p|^2 + \eta^2}.$$

This together with $r^+(\omega) \geq \text{dist}(B', B)$ yields

$$I(x, \tau) \geq \text{dist}(B', B) \int_{S(x, B)} d\omega \int_{r^+(\omega)}^{r^-(\omega)} e^{-\tau r} dr$$

and thus we have

$$\tau I(x, \tau) \geq \frac{\text{dist}(B', B)}{4\pi} \left(\int_{S(x, B)} e^{-\tau r^+(\omega)} d\omega - \int_{S(x, B)} e^{-\tau r^-(\omega)} d\omega \right). \quad (4.2)$$

Define $d_B(x) = \inf\{|x - y| \mid y \in B\}$. First we give an estimate for the second integral in the right hand side of (4.2). Since $d_B(x) = |x - p| - \eta$ and $r^-(\omega) > \sqrt{|x - p|^2 - \eta^2}$, we have

$$\begin{aligned} \int_{S(x, B)} e^{-\tau r^-(\omega)} d\omega &\leq \int_{S(x, B)} e^{-\tau \sqrt{d_B(x)} \sqrt{|x-p|+\eta}} d\omega \\ &\leq 4\pi e^{-\tau d_B(x)} e^{-\tau(\sqrt{d_B(x)} \sqrt{|x-p|+\eta} - d_B(x))}. \end{aligned}$$

Here note that

$$\sqrt{d_B(x)} \sqrt{|x-p|+\eta} - d_B(x) = \frac{2\eta \sqrt{d_B(x)}}{\sqrt{|x-p|+\eta} + \sqrt{d_B(x)}} > 0.$$

Thus we obtain

$$e^{\tau d_B(x)} \int_{S(x, B)} e^{-\tau r^-(\omega)} d\omega \leq 4\pi e^{-A\tau} \quad (4.3)$$

where

$$A = \inf_{x \in B'} \frac{2\eta\sqrt{d_B(x)}}{\sqrt{|x-p|+\eta} + \sqrt{d_B(x)}}.$$

The points are: A is positive and independent of $x \in B'$.

Next we consider the first integral of the right hand side of (4.2). The surface $S(x, B)$ has the parameterization:

$$\omega(r, \theta) = r \cos \theta \mathbf{a}(x) + r \sin \theta \mathbf{b}(x) + \frac{p-x}{|p-x|} \sqrt{1-r^2}, \quad (4.4)$$

where $0 < r < \eta/|x-p|$ and $0 \leq \theta < 2\pi$; $\mathbf{a}(x)$ and $\mathbf{b}(x)$ are unit vectors perpendicular each other and satisfy $\mathbf{a}(x) \times \mathbf{b}(x) = (p-x)/|p-x|$. Since $d\omega = (r/\sqrt{1-r^2})drd\theta$, one can write

$$\int_{S(x,B)} e^{-\tau r^+(\omega)} d\omega = \int_0^{2\pi} d\theta \int_0^{\eta/|x-p|} e^{-\tau r^+(\omega(r,\theta))} \frac{rdr}{\sqrt{1-r^2}}.$$

Here we note that $r^+(\omega(r, \theta))$ is independent of θ . In fact we have

$$r^+(\omega(r, \theta)) = |p-x|\sqrt{1-r^2} - \sqrt{\eta^2 - r^2}|p-x|^2.$$

Thus this yields

$$\int_{S(x,B)} e^{-\tau r^+(\omega)} d\omega \geq 2\pi \int_0^{\eta/|p-x|} e^{-\tau r^+(\omega(r,0))} rdr. \quad (4.5)$$

Since $d_B(x) = |p-x| - \eta$, we obtain

$$\begin{aligned} r^+(\omega(r, 0)) - d_B(x) &= |p-x|\sqrt{1-r^2} - \sqrt{\eta^2 - r^2}|p-x|^2 - |p-x| + \eta \\ &= |p-x|(\sqrt{1-r^2} - 1) + (\eta - \sqrt{\eta^2 - r^2}|p-x|^2) \\ &= r^2|p-x| \left(\frac{1}{\eta + \sqrt{\eta^2 - r^2}|p-x|^2} - \frac{1}{1 + \sqrt{1-r^2}} \right) \\ &\leq \frac{r^2|p-x|}{\eta + \sqrt{\eta^2 - r^2}|p-x|^2} < \frac{r^2|p-x|}{\eta}. \end{aligned} \quad (4.6)$$

Now set $M = \sup_{x \in B'} (|p-x|/\eta)$. Since $r^+(\omega(r, 0)) - d_B(x) \geq 0$, from (4.6) we obtain

$$\begin{aligned} \int_0^{\eta/|x-p|} e^{-\tau r^+(\omega(r,0))} rdr &\geq e^{-\tau d_B(x)} \int_0^{\eta/|p-x|} e^{-\tau B r^2} rdr \\ &= \frac{e^{-\tau d_B(x)}}{2\tau M} (1 - e^{-\tau B(\eta/|p-x|)^2}) \\ &\geq \frac{e^{-\tau d_B(x)}}{2\tau M} (1 - e^{-\tau/B}). \end{aligned} \quad (4.7)$$

From this, (4.2), (4.3) and (4.5) we can conclude that: there exist $\tau_0 > 0$ and $C' > 0$ independent of $x \in B'$ such that, for all $\tau \geq \tau_0$

$$\tau^2 e^{\tau d_B(x)} I(x, \tau) \geq C'.$$

This together with (4.1) yields

$$\tau^4 \int_{B'} v(x)^2 dx \geq (CC')^2 \int_{B'} e^{-2\tau d_B(x)} dx = (CC')^2 e^{2\tau\eta} \int_{B'} e^{-2\tau|x-p|} dx.$$

We have already known (Proposition 3.2 in [6]) that

$$\liminf_{\tau \rightarrow \infty} \tau^2 e^{2\tau d_{B'}(p)} \int_{B'} e^{-2\tau|x-p|} dx > 0. \quad (4.8)$$

This yields

$$\liminf_{\tau \rightarrow \infty} \tau^6 e^{2\tau(d_{B'}(p)-\eta)} \int_{B'} v(x)^2 dx > 0.$$

Since $d_{B'}(p) - \eta = \text{dist}(B', B)$ we conclude that (2.15) is valid.

□

4.2 Proof of Lemma 3.1.

We employ the same notation used in the proof of Lemma 2.1. It suffices to prove (3.12) in the case when $D = B'$. Since

$$\nabla v(x) = -\frac{1}{4\pi} \int_B \frac{1 + \tau|x-y|^2}{|x-y|^3} e^{-\tau|x-y|} f(y)(x-y) dy,$$

we have

$$\begin{aligned} & (4\pi)^2 |\nabla v(x)|^2 \\ &= \int_{B \times B} K(x, y_1) K(x, y_2) e^{-\tau(|x-y_1|+|x-y_2|)} f(y_1) f(y_2) (x-y_1) \cdot (x-y_2) dy_1 dy_2, \end{aligned} \quad (4.9)$$

where

$$K(x, y) = \frac{1 + \tau|x-y|^2}{|x-y|^3}.$$

Define

$$J(x, \tau) = \int_B \frac{(x-y)}{|x-y|^3} e^{-\tau|x-y|} dy.$$

Since it holds that $(x-y_1) \cdot (x-y_2) > 0$ for all $y_1, y_2 \in B$ and $x \in \mathbf{R}^3 \setminus \overline{B}$ and $f(y_1)f(y_2) \geq C^2$ a.e. $y_1, y_2 \in B$, from (4.8) we obtain

$$\begin{aligned} & (4\pi)^2 |\nabla v(x)|^2 \\ & \geq C^2 \int_{B \times B} K(x, y_1) K(x, y_2) e^{-\tau(|x-y_1|+|x-y_2|)} (x-y_1) \cdot (x-y_2) dy_1 dy_2 \\ & \geq C^2 (1 + \tau \text{dist}(D, B)^2)^2 \int_{B \times B} \frac{(x-y_1) \cdot (x-y_2)}{|x-y_1|^3 |x-y_2|^3} e^{-\tau(|x-y_1|+|x-y_2|)} dy_1 dy_2 \\ & = C^2 (1 + \tau \text{dist}(D, B)^2)^2 |J(x, \tau)|^2. \end{aligned} \quad (4.10)$$

One can write

$$\begin{aligned} J(x, \tau) &= \int_{S(x, B)} \omega d\omega \int_{r^+(\omega)}^{r^-(\omega)} e^{-\tau r} dr \\ &= \frac{1}{\tau} \left(\int_{S(x, B)} e^{-\tau r^+(\omega)} \omega d\omega - \int_{S(x, B)} e^{-\tau r^-(\omega)} \omega d\omega \right). \end{aligned} \quad (4.11)$$

Since $r^+(\omega) \geq d_B(x)$, we have

$$\left| \int_{S(x, B)} e^{-\tau r^+(\omega)} \omega d\omega \right| \leq 4\pi e^{-\tau d_B(x)}.$$

The second integral in the right hand side of (4.11) has the bound $O(e^{-\tau d_B(x)} e^{-A\tau})$. These give

$$\tau^2 |J(x, \tau)|^2 = \left| \int_{S(x, B)} e^{-\tau r^+(\omega)} \omega d\omega \right|^2 + O(e^{-A\tau} e^{-2\tau d_B(x)}). \quad (4.12)$$

Using parameterization (4.4) of $S(x, B)$, one has

$$\int_{S(x, B)} e^{-\tau r^+(\omega)} \omega d\omega = \int_0^{2\pi} d\theta \int_0^{\eta/|x-p|} e^{-\tau r^+(\omega(r, \theta))} \omega(r, \theta) \frac{r dr}{\sqrt{1-r^2}}.$$

Since $r^+(\omega(r, \theta))$ is independent of θ and

$$\int_0^{2\pi} \omega(r, \theta) d\theta = \frac{p-x}{|p-x|} \sqrt{1-r^2} 2\pi,$$

we obtain

$$\int_{S(x, B)} e^{-\tau r^+(\omega)} \omega d\omega = 2\pi \frac{p-x}{|p-x|} \int_0^{\eta/|p-x|} e^{-\tau r^+(\omega(r, 0))} r dr.$$

From this together with (4.7) and (4.12) one can conclude that: there exist $\tau_0 > 0$ and $C' > 0$ independent of $x \in B'$ such that, for all $\tau \geq \tau_0$

$$\tau^4 e^{\tau 2d_B(x)} |J(x, \tau)|^2 \geq C'.$$

Thus from (4.10) we obtain

$$\tau^2 \int_{B'} |\nabla v(x)|^2 dx \geq C'' \int_{B'} e^{-2\tau d_B(x)} dx = C'' e^{2\tau \eta} \int_{B'} e^{-2\tau |x-p|} dx,$$

where C'' is a positive constant. Hereafter using (4.8), we obtain (3.12).

□

5 Conclusion and further problems

In this paper we introduce a simple method for some class of inverse obstacle scattering problems that employs the values of the wave field over a *finite* time interval on a known surface surrounding unknown obstacles as the observation data. The wave field is generated by an initial data localized outside the surface and its form is not specified except

for the condition on the support. The method yields information about the location and shape of the obstacles more than the convex hull.

- It would be interesting to apply the method presented in this paper to other time dependent problems in electromagnetism, linear elasticity, classical fluids etc.. Those applications belong to our future plan.

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